§ Tensors

Recall: (Linear Algebra)

dim: $\dim V$ $\dim V + \dim W$ $\dim V \times \dim W$

$$V, W \text{ vector space } (/\mathbb{R}) \rightsquigarrow V^*,\ V \oplus W,\ V \otimes W$$

V^*: dual space; $V \oplus W$: direct sum; $V \otimes W$: tensor product.

Tensor Product: $\dim(V \otimes W) = \dim V \cdot \dim W$.

$$V \otimes W := \left\{ \sum_{i=1}^{k} a_i (v_i \otimes w_i) \,\middle|\, a_i \in \mathbb{R},\ v_i \in V,\ w_i \in W \right\}$$

s.t.
$$\left.\begin{aligned} (a_1 v_1 + a_2 v_2) \otimes w &= a_1 (v_1 \otimes w) + a_2 (v_2 \otimes w) \\ v \otimes (b_1 w_1 + b_2 w_2) &= b_1 (v \otimes w_1) + b_2 (v \otimes w_2) \end{aligned}\right\} \text{"bilinearity" } \cdot \otimes \cdot$$

Equivalently, we view:

$$V^* \otimes W^* \cong \{ \phi : V \times W \to \mathbb{R} \text{ "bilinear" } \}$$

i.e. $\phi(\cdot, w) : V \to \mathbb{R}$ linear for each fixed $w \in W$

$\phi(v, \cdot) : W \to \mathbb{R}$ linear for each fixed $v \in V$.

Recall: $\exists$ natural / canonical pairing

$$\begin{array}{ccc} V \times V^* & \longrightarrow & \mathbb{R} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (v, v^*) & \longmapsto & v^*(v) \end{array}$$

We have for any $v^* \in V^*$, $w^* \in W^*$,

$$V^* \otimes W^* \ni (v^* \otimes w^*)(v, w) := v^*(v) \cdot w^*(w)$$

We can define tensor product of linear maps as follows:

Given linear maps $T: V \to \tilde{V}$, $S: W \to \tilde{W}$.

$$T \otimes S : V \otimes W \longrightarrow \tilde{V} \otimes \tilde{W}$$
$$v \otimes w \longmapsto T(v) \otimes S(w)$$

Ex: Well-defined?

Moral: Any "canonical" (i.e. indep. of choice of basis) constructions for vector spaces can be done fiberwise on vector bundles.

Applying to the tangent bundle TM

$$TM := \coprod_{p\in M} T_pM \xrightarrow{\text{dual}} T^*M := \coprod_{p\in M} T_p^*M$$ cotangent bundle

contravariant

$$T^r_s M := \coprod_{p \in M} \underbrace{(T_pM \otimes \cdots \otimes T_pM)}_{r\text{-times}} \otimes \underbrace{(T_p^*M \otimes \cdots \otimes T_p^*M)}_{s\text{-times}}$$

covariant

∈ (r, s)-tensor bundle over M

E.g.) $T^1_0 M = TM$; $T^0_1 M = T^*M$

Def$^{\text{n}}$: $\mathcal{T}(T^r_sM) := \{ (r,s)\text{-tensors on } M \}$ "$C^\infty(M)$-module"

Some algebraic tensor operations

① tensor product $\otimes$

② "Contraction" (w.r.t. i,j): $C_{ij} : V^{\otimes p} \otimes V^{*\otimes q} \longrightarrow V^{\otimes(p-1)} \otimes V^{*\otimes(q-1)}$

$$C_{ij}(v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^*)$$
$$= v_j^*(v_i)\,(v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_p) \otimes (v_1^* \otimes \cdots \otimes \widehat{v_j^*} \otimes \cdots \otimes v_q^*)$$

E.g.) $C_{1,1} : V \otimes V^* \to \mathbb{R}$; $C_{1,1}(v \otimes v^*) = v^*(v)$

<u>Note</u>: This is just the "trace" on $\mathrm{End}(V) \cong V \otimes V^*$

<u>Ex</u>: Check this! i.e. $(v \otimes v^*)(w) = v^*(w) \cdot v$

③ <u>"Interior Product"</u> (w.r.t $v \in V$)

Given $\alpha \in (V^*)^{\otimes q}$, i.e. $\alpha : \overbrace{V \times \cdots \times V}^{q\text{-times}} \to \mathbb{R}$ multilinear.

define $(\iota_v \alpha) \in (V^*)^{\otimes (q-1)}$ as

$$(\iota_v \alpha)(v_1, \dots, v_{q-1}) := \alpha(v, v_1, \dots, v_{q-1})$$

<u>Pullback of tensors</u>

Given a diffeo. $\phi : M \to N$, we can pullback (p,q)-tensors on N to obtain (p,q)-tensors on M as follow:

$$\phi^*\alpha \xleftarrow{\phi^*} \alpha \quad ((p,q)\text{-tensor})$$

$$\begin{tikzcd} M \arrow[r, shift left, "\phi"] & N \arrow[l, shift left, "\phi^{-1}"] \end{tikzcd} \quad \text{diffeo.}$$

$$\phi^* : \Gamma(T^p_q N) \to \Gamma(T^p_q M)$$

s.t.
- $\phi^*(X) = (\phi^{-1})_* X \quad \forall X \in \Gamma(TN)$ — vector fields
- $(\phi^*\alpha)(X)(x) = \alpha_{\phi(x)}(d\phi_x(X)) \quad \forall \alpha \in \Gamma(T^0_1 N)$, $\forall X \in \Gamma(TM)$, $x \in M$ — 1-forms
- $\phi^*(\alpha \otimes \beta) = \phi^*\alpha \otimes \phi^*\beta \quad \forall$ tensors α, β of any type.

<u>Remarks</u>: (i) $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ for $\phi, \psi \in \mathrm{Diff}$.

(ii) ϕ^* commutes with any contraction.

§ Lie derivative

Given $X \in \mathcal{T}(TM)$, we can define the Lie derivative (w.r.t X)

flow $\{\phi_t\}_t$

$$\mathcal{L}_X : \mathcal{T}(T^p_q M) \longrightarrow \mathcal{T}(T^p_q M)$$

by

$$\mathcal{L}_X \alpha := \frac{d}{dt}\Big|_{t=0} (\phi_t^* \alpha)$$

Properties of $\mathcal{L}_X$

(a) $\mathcal{L}_X f = X(f) = df(X)$, $\forall f \in C^\infty(M)$

(b) $\mathcal{L}_X Y = [X, Y]$ $\quad \forall Y \in \mathcal{T}(TM)$

(c) $\mathcal{L}_X(\alpha \otimes \beta) = (\mathcal{L}_X \alpha) \otimes \beta + \alpha \otimes (\mathcal{L}_X \beta)$ $\quad \forall$ tensors α, β.

(d) $\mathcal{L}_X \circ C = C \circ \mathcal{L}_X$ $\quad \forall$ contraction C

FACT: These 4 properties uniquely characterize $\mathcal{L}_X$.

Reason: Suppose $\exists$ linear map

$$P_X : \mathcal{T}(T^p_q M) \to \mathcal{T}(T^p_q M)$$

satisfying (a) - (d) above. Claim: $P_X = \mathcal{L}_X$.

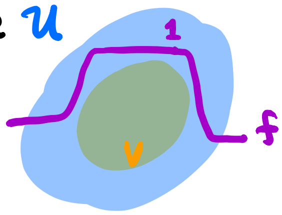
First, we show P_X is a "local" operator:

i.e. Suppose $\alpha, \beta \in \mathcal{T}(T^p_q M)$ s.t. $\alpha|_U \equiv \beta|_U$ on some open $U \subseteq M$.

Then, $(P_X \alpha)|_U \equiv (P_X \beta)|_U$

Why? Choose another open $V \subset\subset \mathcal{U}$, ie $\bar{V} \subset \mathcal{U}$.

$M \supseteq \mathcal{U}$



Fix $f \in C^\infty(M)$ cutoff fcn st

$$\begin{cases} f \equiv 1 & \text{on } V \\ f \equiv 0 & \text{outside } \mathcal{U}. \end{cases}$$

Now, $\alpha|_{\mathcal{U}} \equiv \beta|_{\mathcal{U}} \Rightarrow f\alpha = f\beta$ on M

$$(c) \Rightarrow \underbrace{(P_X f)}_{(a)\ X(f)} \alpha + f(P_X \alpha) = \underbrace{(P_X f)}_{X(f)} \beta + f(P_X \beta) \quad \text{on } M$$

$$\Rightarrow \quad f(P_X \alpha) = f(P_X \beta) \quad \text{on } \mathcal{U}$$

$$\Rightarrow \quad P_X \alpha = P_X \beta \quad \text{on } V$$ $\Rightarrow$ also on $\mathcal{U}$ since V arbitrary.

Application: $L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}$

Q: What is a tensor "really"?

(1,0) - tensor $\leftrightsquigarrow$ vector fields

$\updownarrow$ dual $\qquad\qquad$ $\updownarrow$ dual

(0,1) - tensor $\leftrightsquigarrow$ 1-form

- What about (0,2)-tensors? $\quad$ $C^\infty(M)$-module

(0,2)-tensors $\leftrightsquigarrow$ map $\Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$

bilinear over $C^\infty(M)$

Why? "⇒" Given (0,2)-tensor $\alpha \in \Gamma(T^0_2 M)$. we define

$$\alpha : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$$

s.t. $\alpha(X,Y)(p) = \underbrace{\alpha_p}_{\in\, T_p^*M \otimes T_p^*M}(X_p, Y_p) \quad \forall p \in M$

Note: $\alpha(fX,Y) = f\alpha(X,Y) = \alpha(X,fY)$, $\forall f \in C^\infty(M)$

"$\Leftarrow$" Given a map

$$\Psi : \mathcal{T}(TM) \times \mathcal{T}(TM) \longrightarrow C^\infty(M), \quad C^\infty(M)\text{-bilinear}$$

At each $p \in M$, we define a bilinear map (over $\mathbb{R}$)

$$\alpha_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

as follows, let $X_p, Y_p \in T_pM$, extending to $X, Y \in \mathcal{T}(TM)$ non-unique

$$\Psi(X,Y)(p) =: \alpha_p(X_p, Y_p)$$

Ex: uses bilinearity / $C^\infty(M)$ $\Rightarrow$ well-defined $\because$ indep of the extensions X, Y

Application: $[\cdot,\cdot] : \mathcal{T}(TM) \times \mathcal{T}(TM) \longrightarrow \mathcal{T}(TM)$ is **NOT** a tensor

Digression: $\mathrm{Hom}(V,W) \cong V^* \otimes W$; $(V \otimes W)^* \cong V^* \otimes W^*$

$$\underset{C^\infty(M)\text{-linear}}{\Psi : \mathcal{T}(T^p_q M) \to \mathcal{T}(T^r_s M)} \iff \Psi \in \mathcal{T}\big(\underbrace{(T^p_q M)^* \otimes T^r_s M}_{T^{q+r}_{p+s} M}\big)$$

Since $[fX, gY] = \underbrace{(f\cdot(Xg))Y - (g\cdot(Yf))X}_{\text{NOT } C^\infty(M)\text{-biline}} + fg[X,Y]$

$f, g \in C^\infty(M)$